

On the Domain of Mixing Angles in Three Flavor Neutrino Oscillations

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We clarify the domain needed for the mixing angles in three flavor neutrino oscillations. By comparing the ranges of the transition probabilities as functions of the domains of the mixing angles, we show that it is necessary and sufficient to let all mixing angles be in $[0, \pi/2]$. This holds irrespectively of any assumptions on the neutrino mass squared differences.

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I. INTRODUCTION

With the growing realization that neutrinos might have non-zero masses, studies of neutrino physics data in terms of neutrino oscillations, with three (or more) flavors, are becoming more and more popular.

The easiest way to describe the neutrino oscillations is to use a set of two flavor oscillation schemes. However, since we observe three neutrino flavors in nature, a more natural approach is to use a three flavor oscillation model, and there are reports of different results obtained, when using two or three flavors [1,2].

Much work has been done on neutrino oscillations using two flavor models. When using a three flavor oscillation model, one simplification often used is to make assumptions of the mass squared differences to enable the problem to be treated using much of the results coming from the use of two flavor oscillation models.

In a two flavor oscillation model it is natural to let the mixing angles be in $[0, \pi/4]$. By assuming the mass squared differences to be in certain ranges that allow the three flavor oscillation model to be simplified, using results from the two flavor oscillation model, this domain of the mixing angles is often inherited to the three flavor oscillation models. Several authors that have made such assumptions about the mass squared differences have restricted the mixing angles to $[0, \pi/4]$ [3]. Other authors, however, have found solutions with mixing angles larger than $\pi/4$ when using a three flavor oscillation model [2].

To clarify the situation we here perform a detailed investigation of the domain needed for the vacuum mixing angles without any specific assumptions and show that for a correct analysis to be possible, it is necessary to let

the domain be $0 \leq \theta_i \leq \pi/2$ for all $i = 1, 2, 3$ [4]. This holds even if assumptions on the mass squared differences are made.

Our paper is organized as follows. In Section 2 we describe our formalism. In Section 3 we analyze the domain needed for three flavor oscillations, and in Section 4 we discuss the results.

II. FORMALISM

For the present analysis we will use the plane wave approximation when describing the neutrino propagation. We will describe the neutrino flavor states produced by charge current weak interactions as a mixing of neutrino mass eigenstates. For simplicity we neglect any CP -changing phases. We then write the flavor states $|\nu_\alpha\rangle$, $\alpha = e, \mu, \tau$, as a linear combination of the mass eigenstates $|\nu_i\rangle$, $i = 1, 2, 3$,

$$|\nu_\alpha\rangle = \sum_{i=1}^3 U_{\alpha i}^* |\nu_i\rangle, \quad \alpha = e, \mu, \tau. \quad (1)$$

Here U is a unitary mixing matrix, given by the standard representation of the Cabibbo-Kobayashi-Maskawa mixing matrix as

$$U = \begin{pmatrix} C_2 C_3 & S_3 C_2 & S_2 \\ -S_3 C_1 - S_1 S_2 C_3 & C_1 C_3 - S_1 S_2 S_3 & S_1 C_2 \\ S_1 S_3 - S_2 C_1 C_3 & -S_1 C_3 - S_2 S_3 C_1 & C_1 C_2 \end{pmatrix}, \quad (2)$$

with $C_i \equiv \cos \theta_i$ and $S_i \equiv \sin \theta_i$, $i = 1, 2, 3$. Since there are no CP phases, the matrix elements are real and we have $U_{\alpha a}^* = U_{\alpha a}$ for $\alpha = e, \mu, \tau$ and $a = 1, 2, 3$.

The transition probability for a neutrino changing from ν_α to ν_β is given by

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4 \sum_{i=1}^3 \sum_{j=1, j \neq i}^3 U_{\alpha i} U_{\beta i} U_{\alpha j} U_{\beta j} \sin^2 \frac{\Delta m_{ij}^2 L}{4E}, \quad (3)$$

where $\alpha, \beta = e, \mu, \tau$. Here $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$, E is the relativistic energy of the neutrinos and L is the source-detector distance.

The mass squared differences are not all independent in a three neutrino model, but fulfill by definition the relation

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$$\Delta m_{21}^2 + \Delta m_{32}^2 + \Delta m_{13}^2 = 0. \quad (4)$$

From Eq. (3) it follows that $P_{\alpha\beta} = P_{\beta\alpha}$.

In what follows we shall, without loss of generality, assume that the masses are ordered so that $m_1 \leq m_2 \leq m_3$, and the mass squared differences Δm_{ij}^2 are non-negative when $i > j$.

From the unitarity condition we have three equations that the transition probabilities must fulfill:

$$P_{ee} + P_{e\mu} + P_{e\tau} = 1, \quad (5)$$

$$P_{e\mu} + P_{\mu\mu} + P_{\mu\tau} = 1, \quad (6)$$

$$P_{e\tau} + P_{\mu\tau} + P_{\tau\tau} = 1. \quad (7)$$

Since we want to investigate the mixing angles θ_i , we will for convenience introduce the auxiliary ‘angles’

$$\Phi_1 = \frac{\Delta m_{21}^2 L}{4E} \quad \text{and} \quad \Phi_2 = \frac{\Delta m_{32}^2 L}{4E}. \quad (8)$$

Using Eq. (4), this gives

$$\frac{\Delta m_{31}^2 L}{4E} = \Phi_1 + \Phi_2. \quad (9)$$

The auxiliary angles are both positive by definition. To introduce the auxiliary angles makes sense, because we do not here care about the mass squared differences. The values of Φ_1 and Φ_2 vary with L/E , which is something that is determined by the experimental setup. The mass squared differences only determine how fast the auxiliary angles vary.

We can now write $P_{\alpha\beta}$ as

$$P_{\alpha\beta} = \delta_{\alpha\beta} - 4 \left[f_{\alpha\beta}^1 \sin^2 \Phi_1 + f_{\alpha\beta}^2 \sin^2 \Phi_2 + f_{\alpha\beta}^3 \sin^2 (\Phi_1 + \Phi_2) \right], \quad (10)$$

where $f_{\alpha\beta}^k = f_{\alpha\beta}^k(\theta_1, \theta_2, \theta_3)$ is given by

$$\begin{aligned} f_{\alpha\beta}^1 &= U_{\alpha 1} U_{\beta 1} U_{\alpha 2} U_{\beta 2}, \\ f_{\alpha\beta}^2 &= U_{\alpha 2} U_{\beta 2} U_{\alpha 3} U_{\beta 3}, \\ f_{\alpha\beta}^3 &= U_{\alpha 1} U_{\beta 1} U_{\alpha 3} U_{\beta 3}. \end{aligned} \quad (11)$$

III. ANALYSIS

To determine the necessary and sufficient domain for θ_i , $i = 1, 2, 3$, we first show that it is sufficient to take $\theta_i \in [0, \pi/2]$, and then continue to look whether it is possible to restrict the domain further.

We first notice, that due to unitarity, it is sufficient to look at the diagonal probabilities, P_{ee} , $P_{\mu\mu}$, and $P_{\tau\tau}$. The nondiagonal probabilities $P_{e\mu}$, $P_{e\tau}$, and $P_{\mu\tau}$, can be calculated from the diagonal ones using Eqs. (5)-(7).

The diagonal probabilities are given by

$$P_{\alpha\alpha} = 1 - 4 \sum_{i=1}^3 \sum_{\substack{j=1 \\ i < j}}^3 U_{\alpha i}^2 U_{\alpha j}^2 \sin^2 \frac{\Delta m_{ij}^2 L}{4E}, \quad \alpha = e, \mu, \tau. \quad (12)$$

Since the $U_{\alpha i} U_{\alpha j}$ appear everywhere as $U_{\alpha i}^2 U_{\alpha j}^2$, every $\sin \theta_i$ and $\cos \theta_i$ will appear as squared, causing the signs to disappear. It is therefore always possible to choose an angle between 0 and $\pi/2$ to parameterize the mixing angles. A sufficient domain for their variation is therefore given by $[0, \pi/2]$.

In order to see if it is possible to reduce the domain for θ_i further, we want to find a way to handle the problem that the $P_{\alpha\alpha}$ ’s are functions of five variables, two mass squared differences and three mixing angles. We will do that by fixing Φ_1 and Φ_2 . For every fixed Φ_1 and Φ_2 we then have the $P_{\alpha\alpha}$ ’s as functions of the three mixing angles. This means that we have solved the problem of having too many parameters at the cost of getting a lot of functions instead. For each pair of Φ_1 and Φ_2 there is a function that depends on three parameters, the mixing angles.

Next, it turns out to be practical not to explicitly look for the domain needed for the mixing angles, but instead check if there is any difference between different domains of the angles. If we look at one of the $P_{\alpha\alpha}$ ’s, it will take some range of values for some fixed domain of values of the θ_i ’s. If there are some values it cannot take when the angles are restricted to a smaller domain, the restriction makes a difference. We will therefore look at the range of $P_{\alpha\alpha}$, denoted $R(P_{\alpha\alpha})$.

Considering one probability at a time and without any restrictions on the parameters, the range of $P_{\alpha\alpha}$ is of course $R(P_{\alpha\alpha}) = [0, 1]$. However, for fixed values of Φ_1 and Φ_2 , the range $R(P_{\alpha\alpha})$ can be smaller. To see this, take $L = 0$, giving $\Phi_1 = \Phi_2 = 0$, as an example. This leads to $P_{\alpha\alpha} = 1$ for all possible θ_i ’s, which is obviously a smaller range.

To simplify the forthcoming discussion we introduce

$$P'_{\alpha\alpha} = 4 \left[f_{\alpha\alpha}^1 \sin^2 \Phi_1 + f_{\alpha\alpha}^2 \sin^2 \Phi_2 + f_{\alpha\alpha}^3 \sin^2 (\Phi_1 + \Phi_2) \right]. \quad (13)$$

We do this because all $f_{\alpha\alpha}^k$, $k = 1, 2, 3$, have at least one factor $\sin \theta_i$ for some i , which means that

$$P'_{\alpha\alpha} = 0 \quad \text{when} \quad \theta_1 = \theta_2 = \theta_3 = 0, \quad (14)$$

and we thus obtain a range of the type $[0, x]$, where $x \leq 1$, for all $P'_{\alpha\alpha}$ ’s. We see that the interesting value is the upper limit of $R(P'_{\alpha\alpha})$.

From now on we will always consider Φ_1 and Φ_2 as fixed when calculating $R(P'_{\alpha\alpha})$, making $R(P'_{\alpha\alpha})$ a function of Φ_1 and Φ_2 . This means that we take fixed Φ_1 and Φ_2 and then calculate the range of the function, which now has the three mixing angles as parameters. Each

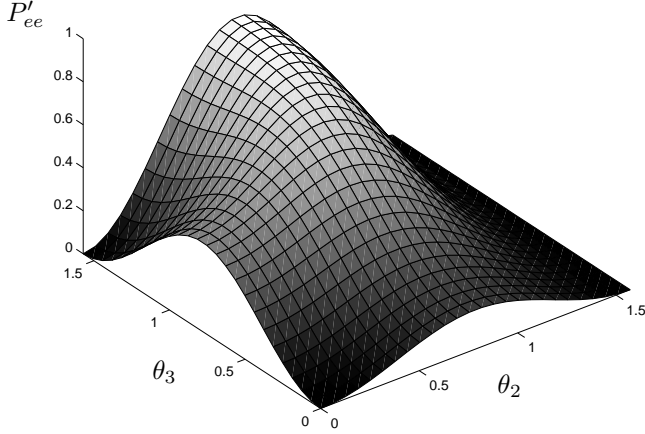


FIG. 1. P'_{ee} , $\Phi_1 = 0.7$ and $\Phi_2 = 2.0$. Note that P'_{ee} does not depend on θ_1 .

choice of Φ_1 and Φ_2 will give a specific range. The upper limit of the range can obviously change for different domains of the θ_i 's. Introduce the notation

$$R(P'_{\alpha\alpha})_{abc} \equiv \{ P'_{\alpha\alpha}(\theta_1, \theta_2, \theta_3) : \theta_1 \in [0, a], \theta_2 \in [0, b], \theta_3 \in [0, c] \} \quad (15)$$

We now calculate $R(P'_{\alpha\alpha})_{abc}$, $abc = \frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2}, \frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4}$. As the $f_k^{\alpha\beta}$'s are 'nice' functions, that vary slowly, this is not a problem to do numerically. Fig. 1 shows an example of how smoothly P'_{ee} varies with θ_i .

The result is, that for some values of Φ_1 and Φ_2 we have

$$R(P'_{\alpha\alpha})_{\frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2}} \neq R(P'_{\alpha\alpha})_{\frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4}}. \quad (16)$$

This means that for some values of Φ_1 and Φ_2 , there are values of the functions $P'_{\alpha\alpha}$ that can be obtained by letting $\theta_i \in [0, \pi/2]$, but not be when $\theta_i \in [0, \pi/4]$.

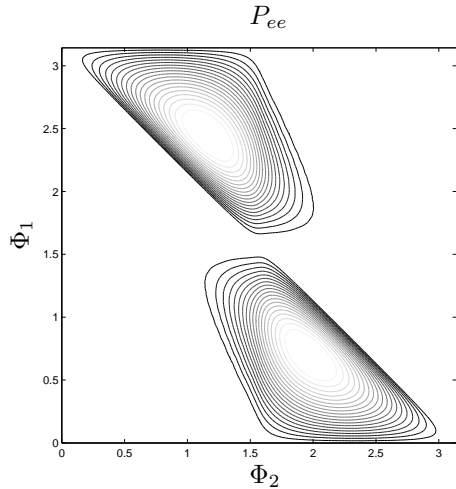


FIG. 2. Plot of $R(P'_{ee})_{\frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2}} - R(P'_{ee})_{\frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4}}$.

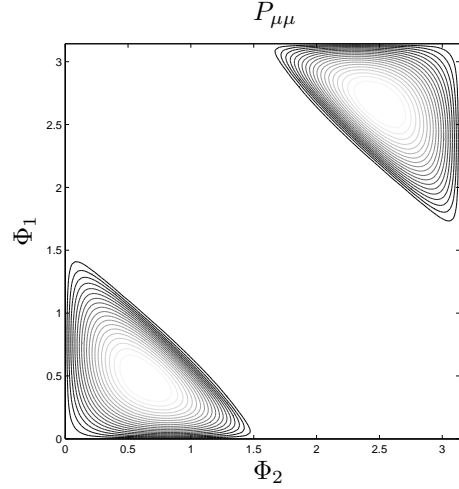


FIG. 3. Plot of $R(P'_{\mu\mu})_{\frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2}} - R(P'_{\mu\mu})_{\frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4}}$.

In the same way, one can check what happens when only one or two mixing angles are restricted to $[0, \pi/4]$. The result is that it is possible to restrict either θ_1 or θ_2 , but not both, to the smaller domain without getting any differences in the ranges of the $P_{\alpha\alpha}$'s.

To visualize this result we define the difference of $R(P'_{abc})$ and $R(P'_{a'b'c'})$ as the difference of the upper limit of the ranges

$$R(P'_{\alpha\alpha})_{abc} - R(P'_{\alpha\alpha})_{a'b'c'} = \max_{\theta_{1,2,3} < a,b,c} (P'_{\alpha\alpha}) - \max_{\theta_{1,2,3} < a',b',c'} (P'_{\alpha\alpha}). \quad (17)$$

We can now make a contour plot of this difference with Φ_1 and Φ_2 on the axes. The result is displayed in Figs. 2-4. In the empty areas of the Figs. there is no difference between the domains, but inside each curve there is a difference that gets larger as we move inwards. The plots have a two-fold rotational symmetry around the point $\Phi_1 = \Phi_2 = \pi/2$.

For example, in $R(P'_{ee})$ we see from Fig. 2, that the maximum difference occurs when $\Phi_1 \simeq 0.7$ and $\Phi_2 \simeq 2$,

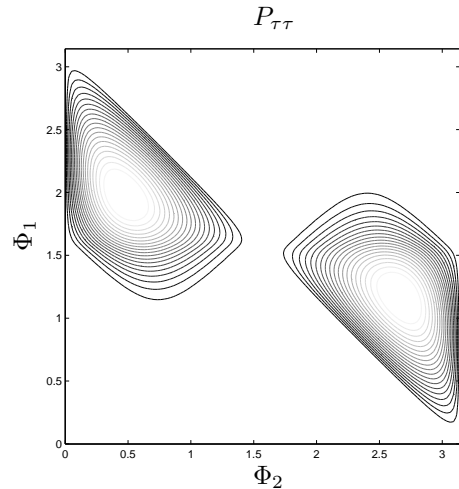


FIG. 4. Plot of $R(P'_{\tau\tau})_{\frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2}} - R(P'_{\tau\tau})_{\frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4}}$.

(or $\Phi_1 \simeq 2.4$ and $\Phi_2 \simeq 1.1$).

If we look at the probability at that point, we have

$$P'_{ee} = 4[(C_2 C_3 S_3 C_2)^2 \sin^2 \Phi_1 + (S_3 C_2 S_2)^2 \sin^2 \Phi_2 + (C_2 C_3 S_2)^2 \sin^2(\Phi_1 + \Phi_2)]. \quad (18)$$

Since this equation only depends on θ_2 and θ_3 , its range can easily be shown in the plot displayed in Fig. 1. It is clear that the range is larger when allowing $\theta_3 > \pi/4$.

To see how large the difference is, we have given the maximum difference for each of the $P'_{\alpha\alpha}$ in Table I. It can readily be seen that the maximum differences are large enough to be of importance.

$P_{\alpha\alpha}$	Maximum difference
P_{ee}	≈ 0.19
$P_{\mu\mu}$	≈ 0.05
$P_{\tau\tau}$	≈ 0.19

TABLE I. Maximum difference between $R(P'_{\alpha\alpha})_{\frac{\pi}{2} \frac{\pi}{2} \frac{\pi}{2}}$ and $R(P'_{\alpha\alpha})_{\frac{\pi}{4} \frac{\pi}{4} \frac{\pi}{4}}$.

For the nondiagonal probabilities the differences are even larger.

IV. DISCUSSION AND CONCLUSION

To see which areas of the plots that are relevant, we look at the definition of Φ_1 and Φ_2 in Eq. (8). The two mass squared differences have fixed physical values (although largely unknown at present). The quantity L/E can be used as a parameter for Φ_1 and Φ_2 . Varying L/E corresponds to a line in the plots in Figs. 2-4 modulo π . An example of how the line may look like is given in Fig. 5.

The slope of the line depends on the ratio between the two mass squared differences and the size of the area it

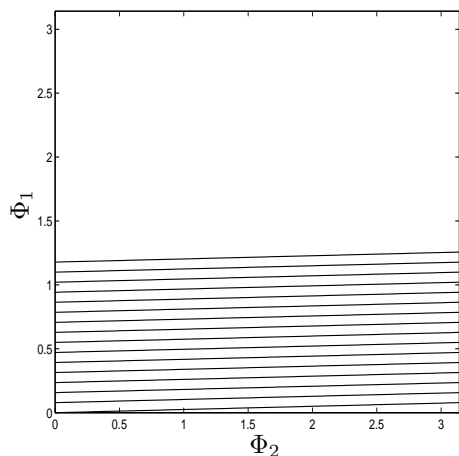


FIG. 5. Example of how the line made up by Φ_1 and Φ_2 as functions of L/E may look like.

covers depends on what values L/E can take. We see that the slope of the line is of little importance when we check if it intersects a marked area in Figs. 2-4. This means that, even when making assumptions of the mass squared differences to simplify the analysis, it is necessary to allow the mixing angles to be in $[0, \pi/2]$.

When considering a line as in Fig. 5, one should remember that the length of the line depends on the range of values that L/E can take in an experiment. In a particular experiment this may be a very short linesegment, but this does not imply that we only have to consider this short line. By making the experiment better, or using other experiments measuring the same probability, the line will be longer and thus any imaginable extension of the experiment should be taken into account when considering the line in Fig. 5. This means that in practice the line will always cover the whole plot.

The analysis above showed that it is possible to diminish the domain of either θ_1 or θ_2 without getting any difference of the ranges of the $P_{\alpha\alpha}$'s. It is, however, not at all clear which of them one should choose to diminish. As different choices can lead to different solutions, both of them should be left to be in the domain $[0, \pi/2]$.

It is possible to show, using the same technique as here, that as soon as one tries to diminish the domain of variation of all three mixing angles from $[0, \pi/2]$, it makes a difference in the ranges.

In conclusion, when using a model for three flavor oscillations as described briefly above, it is necessary and sufficient to allow all of the mixing angles to be in the domain $[0, \pi/2]$ when fitting experimental data, even when making assumptions on the mass squared differences.

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